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NATIONAL DEFENSE RESEARCH COMMITTEE

ARMOR AND ORDNANCE REPORT NO. A-139

ON THE PROPAGATION OF THE PLASTIC DEFORMATION PRODUCED BY AN EXPANDING CYLINDER

by

James S. Koehler

and

Frederick Seitz

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ON THE PROPAGATION OF THE PLASTIC DEFORMATION PRODUCED BY AN EXPANDING CYLINDER

by

James S. Koehler

and

Frederick Seitz

Approved on January 20, 1943
for submission to the Division Chief

Frederick Seitz
F. Seitz, Jr., Member
Division 2

Approved on January 26, 1943
for submission to the Committee

J. E. Burchard
J. E. Burchard, Chief
Division 2, Structural
Defense and Offense

[redacted]

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Preface

The work described in this report is pertinent to the projects designated by the War Department Liaison Officer as CE-5 and CE-6 and to the projects designated by the Navy Department Liaison Officer as NO-11 and NS-109.

This work has been carried out and reported by the University of Pennsylvania as part of its performance under Contract OEMsr-336.

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Copies No. 58 to 60, inclusive, to the Office of the Secretary of the Committee for transmittal to G. W. Lewis, C. E. MacQuigg and V. N. Krivobok.

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ON THE PROPAGATION OF THE PLASTIC DEFORMATION

PRODUCED BY AN EXPANDING CYLINDER

Abstract

In the present paper it is assumed, in accord with Bethe's model for armor penetration, that the plastic deformation produced during armor penetration is similar to that produced by an expanding cylinder. The paper deals with the deformation produced in a plate which is thick enough so that the plate does not get appreciably thicker near the expanding cylindrical hole. In the first portion of the paper the relations between the stresses and the strains are considered and the wave equations which govern the motion of the material when it is rapidly deformed are derived. It is found that in the case of the thick plate an elastic wave diverges radially from the expanding cylinder, that this is followed by a plastic wave and that the elastic and plastic wave velocities do not differ very much. In the second portion of the paper an approximate expression for the displacement is obtained for the case where the deformation is elastic. In the last section of the paper a method of numerically integrating the wave equations is given. The method is applied to a particular numerical example, and the displacement and the shearing strain produced by a uniformly expanding cylinder are calculated and plotted at various times during the expansion. The calculation shows that the shearing strain in both the elastic and the plastic regions increases as we go towards smaller radii. A discontinuity in the shearing strain is found at the boundary between the elastic and plastic regions. The calculation indicates that the compressibility and the density play an important role in determining the magnitude of the stresses developed in a thick plate. A program for future investigation is suggested.

1. Introduction

Bethe[1/] has calculated the stresses and strains produced in armor plate by a projectile under the assumption that the actual three-dimensional problem can be approximated by a treatment involving only two dimensions. During penetration, a pointed shell will produce large displacements of the surrounding material away from the axis of the shell.

1/ All numerical references are to the list of references given at the end of this report.

Bethe assumes that in a thick plate the displacements parallel to the axis of the shell are negligible compared with displacements perpendicular to the axis. The stresses and strains are calculated by assuming that the projectile produces a cylindrical hole in the plate. As the penetration of the shell proceeds, the hole widens. The initial radius of the hole is zero and its final radius is the radius of the shell.

The forces that displace material away from the axis of the shell must accelerate the material, and they must overcome the stresses produced by the deformation. Bethe recognizes that inertial forces will play a role in the problem, but he neglects them in his treatment. Calculations on the propagation of plastic deformation in steel wires[2] yield kinetic energies that in all cases exceed 40 percent of the total energy. It is to be expected that the kinetic energy may represent a smaller fraction of the total energy in the case of cylindrical waves, but one would not expect the kinetic energy to be entirely negligible.

In his calculation Bethe has given a static treatment of the problem. This means that he calculates the stresses and the strains present in the plate at an infinite time after the penetration of the projectile. Since the initially intense disturbance around the hole eventually spreads over a larger area, it is clear that the maximum stresses and strains produced in the armor are underestimated in Bethe's calculation.

G. I. Taylor has also considered the problem of armor penetration.[3] His first paper gives a static discussion of the deformation

produced in a thin plate by an expanding cylinder. Taylor's relations connecting the stresses and the strains differ from those used by Bethe. In his second paper Taylor gives a dynamic treatment of the enlargement of a hole in a thin plate at high speeds; that is, in this paper he includes inertial forces. The assumption is made that the tensile stress which acts in the plane of the plate and perpendicular to the radial direction is zero.

In the present paper we shall discuss the elastic and plastic waves set up in a thick plate by an expanding cylinder. We shall obtain first the necessary wave equations. An approximate expression for the radial displacement in the elastic case will then be found. Next, a method for numerically integrating the wave equations will be described and, finally, some results obtained by numerical calculation will be given.

2. The wave equations

From the symmetry of the problem we see that the displacement produced by an expanding cylinder in a thick plate is entirely radial. The equation of equilibrium can then be written,[4/]

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} , \qquad (1)$$

where u is the displacement, ρ is the density of the material, r and θ are plane polar coordinates, σ_r and σ_θ are tensile stresses in the directions r and θ, and t is the time. If we assume that the displacement is small compared with r then, according to Timoshenko,[5/] we have

$$\varepsilon_r = \frac{\partial u}{\partial r}, \qquad \varepsilon_\theta = \frac{u}{r},$$
$$\varepsilon_z = \gamma_{r\theta} = \gamma_{rz} = \gamma_{\theta z} = 0. \tag{2}$$

Here ε_r, ε_θ and ε_z are the tensile strains in the directions, r, θ and z, respectively; $\gamma_{r\theta}$, γ_{rz} and $\gamma_{\theta z}$ are the shearing strains; and the z-axis is the axis of the shell. Equations (2) hold only for the case where the displacement is completely radial. It might be well to point out that actually u is not small compared with r near the axis of the shell -- that is, near the z axis. No treatment has as yet been given that deals correctly with this region. We shall use Eqs. (2).

(a) Elastic deformation. -- A wave equation in the variable u can be obtained if we assume relations between the stresses and the strains. If the deformation is elastic, the relations are given by Hooke's law. Thus,

$$\sigma_\theta - \sigma_r = 2G(\varepsilon_\theta - \varepsilon_r), \qquad \sigma_z - \sigma_\theta = -2G\varepsilon_\theta,$$
$$\sigma_r - \sigma_z = 2G\,\varepsilon_r, \qquad \sigma_r + \sigma_\theta + \sigma_z = 3K(\varepsilon_r + \varepsilon_\theta), \tag{3}$$

where G is the modulus of rigidity and K is the modulus of volume expansion. Note that ε_z does not appear in Eqs. (3); in the particular problem considered here, ε_z is zero. If we use Eqs. (2) and substitute values obtained for the stresses from Eqs. (3) into Eq. (1), we obtain the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \left(\frac{3K + 4G}{3\rho}\right)\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right). \tag{4}$$

The velocity of propagation of these elastic waves is

$$c_1 = \sqrt{(3K + 4G)/3\rho}. \tag{5}$$

If we calculate c_1 for steel for the case where $G = 5.40 \times 10^{10}$ lb/ft sec^2, $K = 1.046 \times 10^{11}$ lb/ft sec^2 and $\rho = 480.7$ lb/ft^3, we find that $c_1 = 19{,}170$ ft/sec.

According to Mohr's criterion[6] the material at a specified point will experience plastic deformation if the largest component of the shearing stress at the point exceeds the limit $\frac{1}{2}\lambda$, where $\underline{\lambda}$ is the yield stress obtained in tensile tests made on long thin wires. Huber, Henckey and von Mises[7] postulate that plastic deformation of a polycrystalline material begins when the principal stresses become large enough to satisfy the relation,

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\lambda^2 . \tag{6}$$

The conditions of Mohr and von Mises do not differ very much. The worst agreement is obtained in the case of pure shear. In this case the largest component of the shearing stress found by using Mohr's criterion is 15.5 percent lower than the value obtained by using the von Mises condition. We shall use Mohr's condition to determine whether or not the material at a given point is experiencing plastic deformation.

(b) Plastic deformation. -- We shall assume that the following relations are valid for material that is being plastically deformed:

$$\begin{aligned}
\varepsilon_r &= \frac{1}{D}\left[\sigma_r - \alpha(\sigma_\theta + \sigma_z)\right] + \left(\frac{1}{E} - \frac{1}{D}\right)\sigma_{r_0} - \left(\frac{\nu}{E} - \frac{\alpha}{D}\right)\left[\sigma_{\theta_0} + \sigma_{z_0}\right], \\
\varepsilon_\theta &= \frac{1}{D}\left[\sigma_\theta - \alpha(\sigma_z + \sigma_r)\right] + \left(\frac{1}{E} - \frac{1}{D}\right)\sigma_{\theta_0} - \left(\frac{\nu}{E} - \frac{\alpha}{D}\right)\left[\sigma_{z_0} + \sigma_{r_0}\right], \\
\varepsilon_z &= \frac{1}{D}\left[\sigma_z - \alpha(\sigma_r + \sigma_\theta)\right] + \left(\frac{1}{E} - \frac{1}{D}\right)\sigma_{z_0} - \left(\frac{\nu}{E} - \frac{\alpha}{D}\right)\left[\sigma_{r_0} + \sigma_{\theta_0}\right].
\end{aligned} \tag{7}$$

In Eqs. (7), σ_{r_0}, σ_{θ_0} and σ_{z_0} are the stresses required to initiate plastic deformation. The constants E and ν are Young's modulus and the Poisson ratio. The constants D and α describe the plastic behavior of the material; the way in which they are determined from experimental data will become clear as we proceed. We shall wait until later to discuss the validity and meaning of these equations. Adding Eqs. (7) we obtain

$$\varepsilon_r + \varepsilon_\theta + \varepsilon_z = \frac{1-2\alpha}{D}[\sigma_r + \sigma_\theta + \sigma_z] + \left(\frac{1-2\nu}{E} - \frac{1-2\alpha}{D}\right)[\sigma_{r_0} + \sigma_{\theta_0} + \sigma_{z_0}]. \quad (8)$$

In the elastic case,

$$\varepsilon_r + \varepsilon_\theta + \varepsilon_z = \frac{1}{3K}[\sigma_r + \sigma_\theta + \sigma_z] = \frac{1-2\nu}{E}[\sigma_r + \sigma_\theta + \sigma_z], \quad (9)$$

where $1/K$ is the compressibility. <u>We shall assume that Eq. (9) is also valid for a material which is being plastically deformed.</u>[8/] Since Eq. (8) must be the same as Eq. (9), we have

$$3K = \frac{D}{1-2\alpha}. \quad (10)$$

This equation can be used to determine α once D is known. D can be obtained from the stress-strain curve of the material in tension. Equations (7) assume that this stress strain curve is of the form shown in Fig. 1. The slope of the initial elastic portion of the curve is E. The slope of the stress strain-curve in the plastic region above the knee is D. Equations (7) are, of course, valid only if σ_r, σ_θ and σ_z are principal stresses.

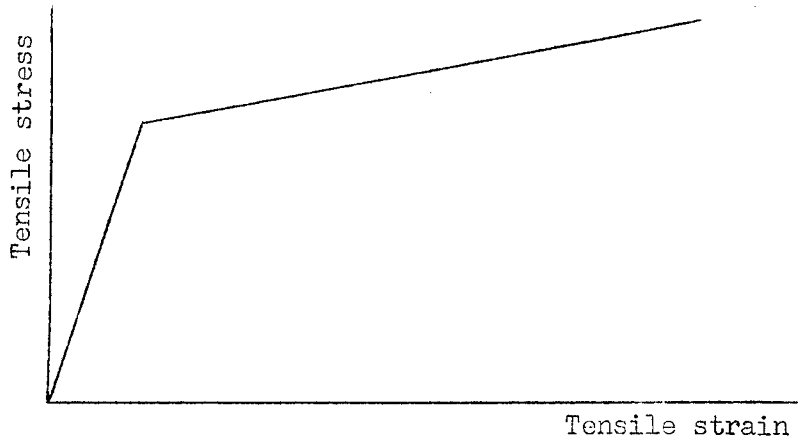

Fig. 1. The form of the stress-strain curve assumed in Eqs. (7).

Ros and Eichinger, Nadai, and others[9] have shown that relations of the type of Eqs. (7) are in accord with experimental tests made on polycrystalline materials. The relations are so constructed that the directions of the principal strains coincide with the directions of the principal stresses. In addition, the equations are so arranged that the figure consisting of Mohr's three principal strain circles remains continuously similar geometrically to the figure made up of the three principal stress circles (see Nadai).[9] Our relations are somewhat more general than theirs since they took K to be infinite -- α is then 0.5 by Eq. (10). The fact that K is not infinite is of importance in the problems treated in the present report.

Let us consider the plastic deformation which occurs in a thick plate. For this case we have:

$$\sigma_{z_0} = \nu(\sigma_{r_0} + \sigma_{\theta_0}), \tag{11}$$

Equations (3) hold at the boundary between the elastic and plastic regions and, since we still assume that the displacement is radial in

the plastic region, Eq. (2) is still valid there. Using the last of Eqs. (2) we can write Eqs. (7) as

$$\begin{aligned}
\sigma_\theta - \sigma_r &= 2F[\varepsilon_\theta - \varepsilon_r] + 2(G - F)[\varepsilon_{\theta_o} - \varepsilon_{r_o}], \\
\sigma_z - \sigma_\theta &= -2F\varepsilon_\theta - 2(G - F)\varepsilon_{\theta_o}, \\
\sigma_r - \sigma_z &= +2F\varepsilon_r + 2(G - F)\varepsilon_{r_o},
\end{aligned} \tag{12}$$

where ε_{θ_o} and ε_{r_o} are the strains present when the material begins to deform plastically, and where

$$F = \frac{D}{2(1 + \alpha)}$$

and

$$G = \frac{E}{2(1 + \nu)},$$

$\underline{G}$ being the elastic shear modulus. In addition we have from Eq. (9), since $\varepsilon_z = 0$,

$$\sigma_r + \sigma_\theta + \sigma_z = 3K[\varepsilon_\theta + \varepsilon_r]. \tag{13}$$

The wave equation for plastic waves is obtained by using Eqs. (1), (2), (12) and (13) in the same way that Eqs. (1), (2) and (3) were used to obtain the elastic wave equation. The wave equation which results is

$$\frac{\partial^2 u}{\partial t^2} = \left(\frac{3K + 4F}{3\rho}\right)\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) - \frac{2(G - F)\gamma_o}{\rho r}, \tag{14}$$

where $\gamma_o = \varepsilon_{\theta_o} - \varepsilon_{r_o}$. The plastic wave velocity is

$$c_2 = \sqrt{(3K + 4F)/3\rho}. \tag{15}$$

Using $F/G = 1/6.94$ for steel, we find that $c_2 = 15,460$ ft/sec. The ratio c_2/c_1 is therefore 0.80. For steel of the quality used in armor plate $\lambda = 4.61 \times 10^8$ lb/ft sec^2, $\gamma_0 = 4.27 \times 10^{-3}$ and $2\gamma_0(G - F)/\rho = 8.20 \times 10^5$ ft^2/sec^2.

(c) Plastic deformation when the slope of the stress-strain curve is some function of the largest shearing strain. -- We can generalize our wave equation by assuming that the material follows a stress-strain curve which can be introduced into the equations as an arbitrary function. If we suppose that the slope of our stress strain curve is some function of the largest shearing strain, then the equations relating the stresses and the strains may be written in a form analogous to Eqs. (3) as

$$\begin{aligned}
\sigma_\theta - \sigma_r &= 2(\varepsilon_\theta - \varepsilon_r)\, f(\varepsilon_\theta - \varepsilon_r), \\
\sigma_z - \sigma_\theta &= -2\varepsilon_\theta\, f(\varepsilon_\theta - \varepsilon_r), \\
\sigma_r - \sigma_z &= 2\varepsilon_r\, f(\varepsilon_\theta - \varepsilon_r), \\
\sigma_r + \sigma_\theta + \sigma_z &= 3K[\varepsilon_r + \varepsilon_\theta].
\end{aligned} \tag{16}$$

Ros and Eichinger have shown experimentally that the D and F of our Eqs. (7) and (12) actually are not constant but depend somewhat on the variable quantity, $\left[(\varepsilon_\theta - \varepsilon_r)^2 + (\varepsilon_z - \varepsilon_\theta)^2 + (\varepsilon_r - \varepsilon_z)^2\right]$. We shall assume that only the largest shearing stress plays an important role. This assumption is related to the results of Ros and Eichinger in the same manner as Mohr's condition for the onset of plastic deformation is to the Huber, Henckey and von Mises condition.

The function $f(\varepsilon_\theta - \varepsilon_r)$ may be determined from static experiments in the following way.* In general, the equations relating the principal strains and the principal stresses can be written,

$$\sigma_1 - \sigma_2 = 2(\varepsilon_1 - \varepsilon_2)f(\varepsilon_1 - \varepsilon_3),$$
$$\sigma_2 - \sigma_3 = 2(\varepsilon_2 - \varepsilon_3)f(\varepsilon_1 - \varepsilon_3),$$
$$\sigma_3 - \sigma_1 = 2(\varepsilon_3 - \varepsilon_1)f(\varepsilon_1 - \varepsilon_3)$$

and

$$\sigma_1 + \sigma_2 + \sigma_3 = 3K(\varepsilon_1 + \varepsilon_2 + \varepsilon_3),$$

where $\varepsilon_1 - \varepsilon_3$ is the largest shearing strain. These relations will then be valid in the case of a tensile test on a long thin wire extending in the x_1-direction. In this case,

$$\sigma_2 = \sigma_3 = 0$$

and, hence, by the second of our general equations,

$$\varepsilon_2 = \varepsilon_3.$$

The equations relating the principal strains and principal stresses become

$$\sigma_1 = 2(\varepsilon_1 - \varepsilon_2)f(\varepsilon_1 - \varepsilon_2), \qquad \sigma_1 = 3K(\varepsilon_1 + 2\,\varepsilon_2).$$

Eliminating ε_2 from these equations, we get

$$f\left(\frac{3}{2}\,\varepsilon_1 - \frac{\sigma_1}{6K}\right) = \frac{\sigma_1}{2\left(\frac{3}{2}\,\varepsilon_1 - \frac{\sigma_1}{6K}\right)}\,.$$

---

* This method was suggested by M. P. White and H. F. Bohnenblust.

Thus by measuring σ_1 and ε_1 in a tensile test and by plotting the function on the right as a function of $\left(\frac{3}{2}\varepsilon_1 - \frac{\sigma_1}{6K}\right)$, the function $\underline{f}$ can be found. If $\underline{K}$ can be taken as infinite -- that is, if the substance is incompressable -- then the foregoing equation becomes,

$$f\left(\frac{3}{2}\varepsilon_1\right) = \frac{\sigma_1}{3\varepsilon_1} .$$

Equations (16) are so constructed that the directions of the principal strains coincide with the directions of the principal stresses. In addition, the equations are so arranged that the figure consisting of Mohr's three principal strain circles remains continuously similar geometrically to the figure made up of the three principal stress circles (see Nadai).[9/]

Proceeding as before, using Eqs. (1), (2) and (16), we find that the wave equation for a material in which the slope of the stress-strain curve depends on the largest shearing strain is

$$\frac{\partial^2 u}{\partial t^2} = \left[\frac{K + \frac{4}{3}f - \frac{2}{3}(2\varepsilon_r - \varepsilon_\theta)f'}{\rho}\right]\frac{\partial^2 u}{\partial r^2} + \left[\frac{K + \frac{4}{3}f + \frac{2}{3}(2\varepsilon_r - \varepsilon_\theta)f'}{\rho}\right]\left[\frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right] . \quad (17)$$

In this equation $\underline{f'}$ represents the derivative of the function $\underline{f}$ with respect to its argument $(\varepsilon_\theta - \varepsilon_r)$. Equation (17) is complicated because the coefficients of the partial derivatives depend on $\underline{u}$. Von Kármán[10/] has discussed the propagation of plastic deformation in wires when the stress-strain curve is given as a function to be determined by experiment. We have not been able to apply von Kármán's method of integration to Eq. (17).

(d) Plastic deformation in a region where the shearing strain decreases with the passage of time. -- After the conical head of the projectile has passed completely through the armor plate, the stresses in the vicinity of the cylindrical hole will begin to decrease. We shall next obtain a wave equation appropriate for this region. If a metal is plastically deformed, moving from A to B to C on the stress-strain curve shown in Fig. 2, and if the largest shearing stress is

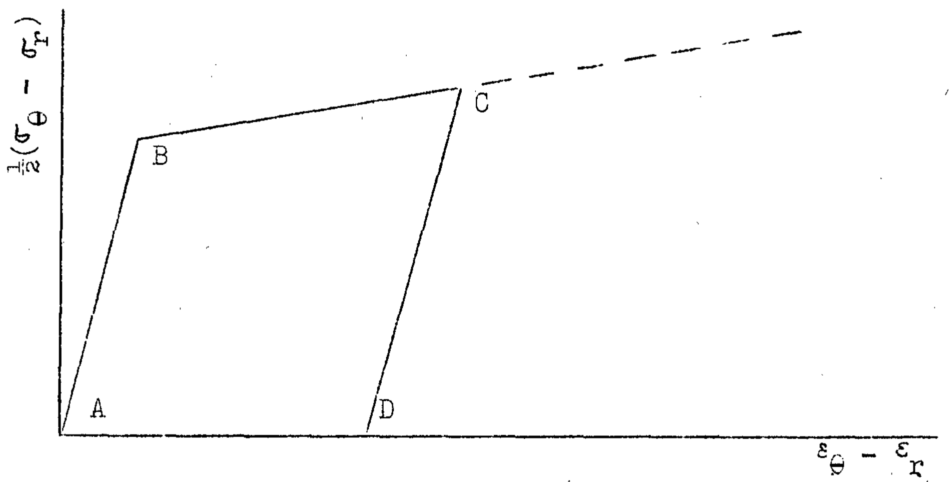

Fig. 2. The stress-strain curve for plastically deformed material including a region CD when the shearing stress is decreasing. Note that CD is parallel to the elastic portion of the curve AB.

then decreased, the shearing strain also decreases; the stress and strain move down along the straight line CD. The line CD is parallel to the elastic portion of the stress-strain curve; that is, CD is parallel to AB. If ε_{r_1} and ε_{θ_1} are the strains at point C of the curve and if γ_1 is the shearing strain at that point, then the equations relating stress and strain along the line CD are

$$\begin{aligned} &\sigma_\theta - \sigma_r = 2G(\varepsilon_\theta - \varepsilon_r) - 2(G-F)(\gamma_1 - \gamma_0), && \sigma_z - \sigma_\theta = -2G\varepsilon_\theta + 2(G-F)(\varepsilon_{\theta_1} - \varepsilon_{\theta_0}), \\ &\sigma_r - \sigma_z = +2G\varepsilon_r - 2(G-F)(\varepsilon_{r_1} - \varepsilon_{r_0}), && \sigma_r + \sigma_\theta + \sigma_z = 3K(\varepsilon_r + \varepsilon_\theta). \end{aligned} \tag{18}$$

The wave equation obtained by using these equations is

$$\frac{\partial^2 u}{\partial t^2} = \left(\frac{3K + 4G}{3\rho}\right)\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) + \frac{2(G - F)(\gamma_i - \gamma_o)}{\rho r}. \qquad (19)$$

The wave velocity is just the velocity of the elastic waves.

3. <u>The elastic case</u>

In order to obtain some idea of the nature of the functions that appear in the solutions of the wave equations, we have attempted to solve a problem involving only elastic waves. Consider a thick plate of infinite extent, through the center of which passes a cylindrical hole of radius b. We shall assume that external forces act against the surface of the hole so that its radius increases with the passage of time. The radial displacement at b is supposed to vary with time in the following fashion:

$$\begin{aligned} u(b) &= 0, && \text{for } t < 0; \\ u(b) &= v_o t, && \text{for } 0 < t < T; \qquad (20) \\ u(b) &= v_o T, && \text{for } T < t. \end{aligned}$$

where v_o and T are constants. Since the deformations are taken to be elastic, the wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c_1^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right], \qquad (4)$$

Bethe[11/] has given a theoretical discussion of a very similar problem. We have extended his treatment to higher orders of approximation, and we have used the method to obtain an expression for u that is valid in the portion of the wave sent out while the cylinder is expanding.

We find that

$$
\begin{aligned}
u = \frac{v_0 b t}{r} + \frac{\pi v_0 b}{c_1} \Bigg\{ & \frac{b}{2c_1 t}\left[\frac{r}{b}\left(1+\frac{r^2}{4c_1^2t^2}+\frac{r^4}{8c_1^4t^4}+\frac{5r^6}{64c_1^6t^6}+\cdots\right)-\frac{b}{r}\left(1+\frac{b^2}{4c_1^2t^2}+\frac{b^4}{8c_1^4t^4}+\frac{5b^6}{64c_1^6t^6}+\cdots\right)\right] \\
& -\frac{b^2 r \log\frac{r}{b}}{2c_1^3t^3}\left[2+\frac{3(r^2+b^2)}{c_1^2t^2}+\frac{15(r^4+3r^2b^2+b^4)}{4c_1^4t^4}+\cdots\right] \\
& +\frac{\pi b^2 r}{c_1^3t^3}\left[\frac{3(r^2-b^2)}{4c_1^2t^2}+\frac{5(r^4-b^4)}{2c_1^4t^4}+\cdots\right] \\
& +\frac{b^2}{c_1^2t^2}\Bigg[+2\left(\frac{b^2}{rc_1t}-\frac{r^2}{bc_1t}\right)\left(Z'(3)-\log\frac{2c_1t}{b}\right) \\
& \qquad +\frac{3b}{c_1t}\left(\frac{2b^3}{rc_1^2t^2}-\frac{(r^3+b^3)}{bc_1^2t^2}\right)\left(Z'(5)-\log\frac{2c_1t}{b}\right) \\
& \qquad +\frac{15}{4}\left(\frac{5b^6}{rc_1^5t^5}-\frac{(br^5+3b^3r^3+b^5r)}{bc_1^5t^5}\right)\left(Z'(7)-\log\frac{2c_1t}{b}\right) \\
& \qquad +\cdots\Bigg] \\
& +\cdots\Bigg\}
\end{aligned} \tag{21}
$$

In this equation $Z'(a) = \frac{d[\log\Gamma(a)]}{da}$, where $\Gamma(a)$ is the gamma function evaluated for argument <u>a</u>. The equation is valid if $c_1(t - T) < r < c_1t$. The complexity of Eq. (21) indicates that it would be difficult to find an analytical expression for the displacement in the case where both elastic and plastic waves are present. The situation would then be more complicated than it is

here since the solutions obtained in the elastic and plastic regions must be fitted together at the boundary between the two regions.

4. Numerical calculations

In our opinion a qualitative understanding of the waves produced by an expanding cylinder is most easily obtained by a numerical integration of the wave equations. In this section a method for numerically integrating the wave equations is described, and some results obtained by numerical calculation are presented.

Let us assume that the cylindrical hole is expanded rapidly enough to cause plastic deformation. In this case both an elastic and a plastic wave are produced. The elastic wave runs out ahead of the plastic wave just as it does when a large force tending to lengthen a long wire is suddenly applied to one end of the wire. The wave equations which must be considered are:

$$\frac{\partial^2 u}{\partial t^2} = c_1^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} \right], \qquad \text{(Elastic)} \qquad (4)$$

$$\frac{\partial^2 u}{\partial t^2} = c_2^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} \right] - \frac{2\gamma_0 (G - F)}{\rho r}. \qquad \text{(Plastic)} \qquad (14)$$

These two differential equations can be written approximately as finite difference equations. To do this we shall consider only the displacements at regularly spaced instants of time and at regularly spaced radii. The time interval will be called Δ; the space interval in the elastic case we shall take to be $c_1\Delta$; and the space interval in the plastic case will be $c_2\Delta$. We shall denote by u_{ij} the displacement at the i th instant of time and at the radius r_j. In this notation

the differential equation for the elastic case becomes

$$\frac{u_{(i+1)j}-2u_{ij}+u_{(i-1)j}}{\Delta^2}=c_1^2\left[\frac{u_{i(j+1)}-2u_{ij}+u_{i(j-1)}}{c_1^2\Delta^2}+\frac{1}{r_j}\left(\frac{u_{i(j+1)}-u_{i(j-1)}}{2c_1\Delta}\right)-\frac{u_{ij}}{r_j^2}\right].$$

Solving this equation for $u_{(i+1)j}$, we find for the elastic case,

$$u_{(i+1)j}=-u_{(i-1)j}+u_{i(j+1)}+u_{i(j-1)}+\frac{c_1\Delta}{2r_j}\left(u_{i(j+1)}-u_{i(j-1)}\right)-\frac{(c_1\Delta)^2u_{ij}}{r_j^2}. \quad (22)$$

Similarly in the plastic region we find

$$u_{(i+1)j}=-u_{(i-1)j}+u_{i(j+1)}+u_{i(j-1)}+\frac{c_1\Delta}{r_j}\left(u_{i(j+1)}-u_{i(j-1)}\right)-\frac{(c_2\Delta)^2u_{ij}}{r_j^2}-\frac{P\Delta^2}{r_j}, \quad (23)$$

where $P = 2\gamma_0(G - F)/\rho$. The method of calculation can be understood with the aid of Fig. 3. In this figure the region that lies above the solid line represents the elastic region; the region below this line represents the plastic region.

Since the maximum velocity of any wave in the material is c_1 and since the material is undisturbed until the time $t = 0$, it is evident that the displacements in the region above the line $r - b = c_1t$ are zero. We shall assume that

$$\begin{aligned} u(b) &= 0, \quad \text{for } t < 0; \\ u(b) &= v_0t, \quad \text{for } t > 0. \end{aligned} \quad (24)$$

Thus, the displacements u_{io} are also known. According to Eqs. (22) and (23) a particular displacement can be calculated if four neighboring displacements are known. The position of these displacements on the diagram is simply related to the position of the displacement

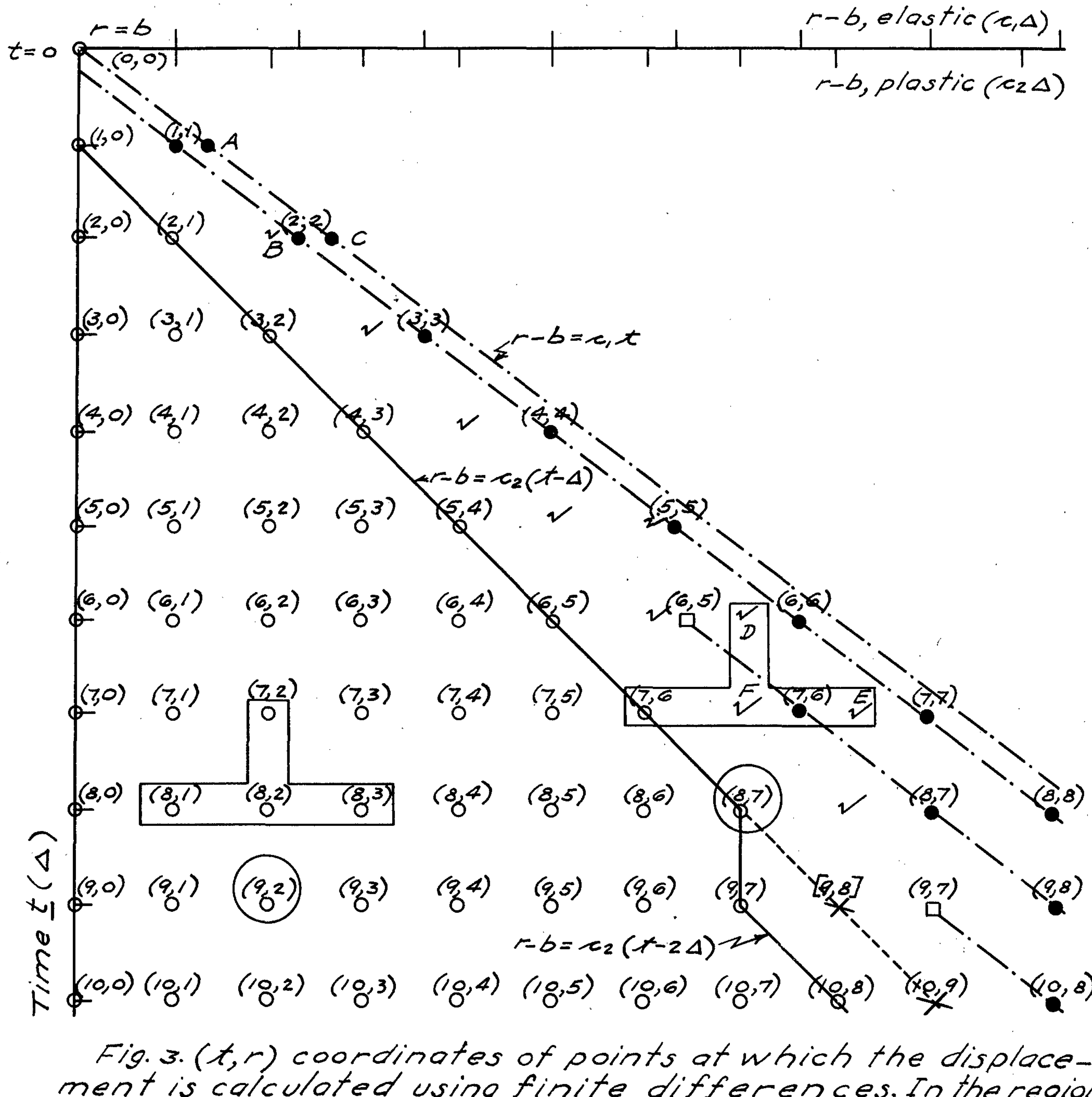

Fig. 3. (t,r) coordinates of points at which the displacement is calculated using finite differences. In the region where no points are given the displacement is zero. In the region above the solid diagonal line the deformation is elastic. In the region below the solid diagonal line the deformation is plastic.

being calculated. For example, if one wishes to calculate u_{92} (see Fig. 3), the displacements required lie in an inverted T just above u_{92}. The shearing strain at any point i,j is given by

$$\gamma_{ij} = \frac{u_{ij}}{r_j} - \frac{u_{i(j+1)} - u_{i(j-1)}}{r_{j+1} - r_{j-1}} . \qquad (25)$$

There are several ways in which the calculation may be performed; we have found the displacements as follows. The elastic displacements are calculated first. The value of u_{11} is found by requiring that the average shearing strain between the point 1,1 and the point A of our diagram must be equal to the shearing strain γ_0 which is required to produce plastic deformation. Thus,

$$\gamma_0 = \frac{u_{11}}{r_b + \Delta c_2} + \frac{u_{11} - 0}{\Delta(c_1 - c_2)} .$$

The displacement u_{22} can then be calculated at once because, of the four displacements necessary for its calculation, only u_{11} differs from zero. Knowing u_{22} we can calculate u_{33}, and so on. Since the elastic wave travels faster than the plastic wave, it soon becomes necessary to introduce another diagonal row of displacements in the elastic region. We have adopted the practice of inserting an additional row whenever the difference in radius between the smallest elastic radius at a particular time differs from the largest plastic radius by more than $\Delta(c_1 + c_2)$. Suppose that the displacement at the top of a new diagonal row

is u_{ab}. Then u_{ab} is found by making the average shearing strain between the points (a,b) and (a,b + 1) equal to γ_o. For instance, u_{65} in the elastic region is given by

$$\gamma_o = \frac{u_{65} - u_{66}}{\Delta c_1} + \frac{u_{65} + u_{66}}{r_5 + r_6} .$$

The other members of the new diagonal row u_{76}, u_{87} and so forth, are then calculated successively using the finite difference equation, Eq. (22).

After the displacements in the elastic region have been evaluated for times between t = 0 and any given later time, the displacements in the plastic region can be calculated for the same range of time. Consider first the displacements along the line $r - b = c_2(t - \Delta)$. In this diagonal row, u_{21} can be found immediately in terms of $u_{10(plastic)}$ and $u_{11(elastic)}$. The displacement u_{32} cannot be evaluated at once because the elastic displacement u_B is not known (see Fig. 3). The same situation occurs all along this diagonal row. For example, suppose that we have found u_{76} and now wish to evaluate u_{87}; the displacement u_{87} can be found only if we know u_D, u_E and u_F. We have evaluated the displacements at points such as B, D, E and F by fitting the nearest elastic displacements to a linear relation between displacement and radius. The linear relation found in this way is then used to calculate the desired displacements. For example, it is assumed that the displacement increases linearly with distance as we go from point C to point 2,2. The displacements $u_C = 0$ and u_{22} are used to determine the constants in the linear relation and u_B is

then calculated using the linear relation. Similarly u_{77} and u_{76} determine the linear equation that is used to find u_F and u_E. The displacements in additional diagonal rows lying successively farther from the first diagonal plastic row are then calculated until all of the plastic displacements between $t = 0$ and the previously specified time are known.

Since the energy density associated with any particular place in our waves decreases as they spread outwards, it is not surprising to find that the shearing strain at the outer edge of the plastic wave decreases with time. Thus, if we evaluate the average shearing strain over the interval between the smallest elastic radius and the largest plastic radius, this strain is for small times larger than γ_0 but eventually decreases until it equals γ_0. When this shearing strain becomes smaller than γ_0, the first diagonal plastic row must be replaced by a diagonal elastic row. We shall show how the replacement is made using the diagram. Let us suppose that we have calculated $u_{87(\text{plastic})}$ and $u_{98(\text{plastic})}$. In our calculations in the elastic region we have started a new diagonal elastic row at $(10,8)_{(\text{elastic})}$, since there the difference between the smallest elastic radius $r_{10,9(\text{elastic})}$ and the largest plastic radius $r_{10,9(\text{plastic})}$ has become greater than $\Delta(c_1 + c_2)$. We shall suppose that further calculation shows that the shearing strain between $(8,7)_{(\text{elastic})}$ and $(8,7)_{(\text{plastic})}$ is greater than γ_0, while the shearing strain between $(9,8)_{(\text{elastic})}$ and $(9,8)_{(\text{plastic})}$ is less than γ_0. We therefore discard our value for $u_{98(\text{plastic})}$ and move the edge of the plastic region back to the line

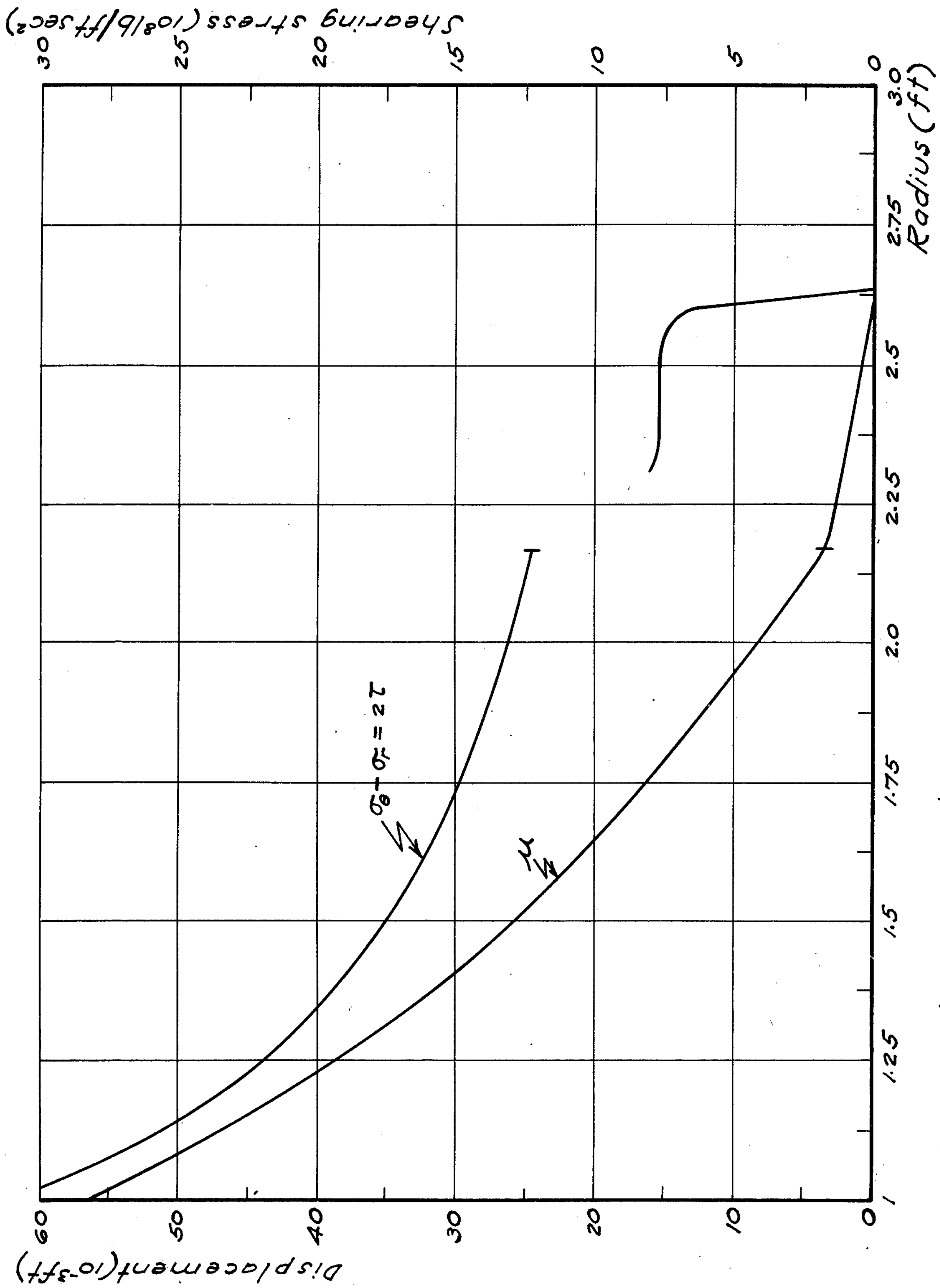

Fig. 4. Displacement and stress produced in a thick plate by a uniformly expanding cylinder. Expansion has been going on for 1.1×10^{-4} sec.

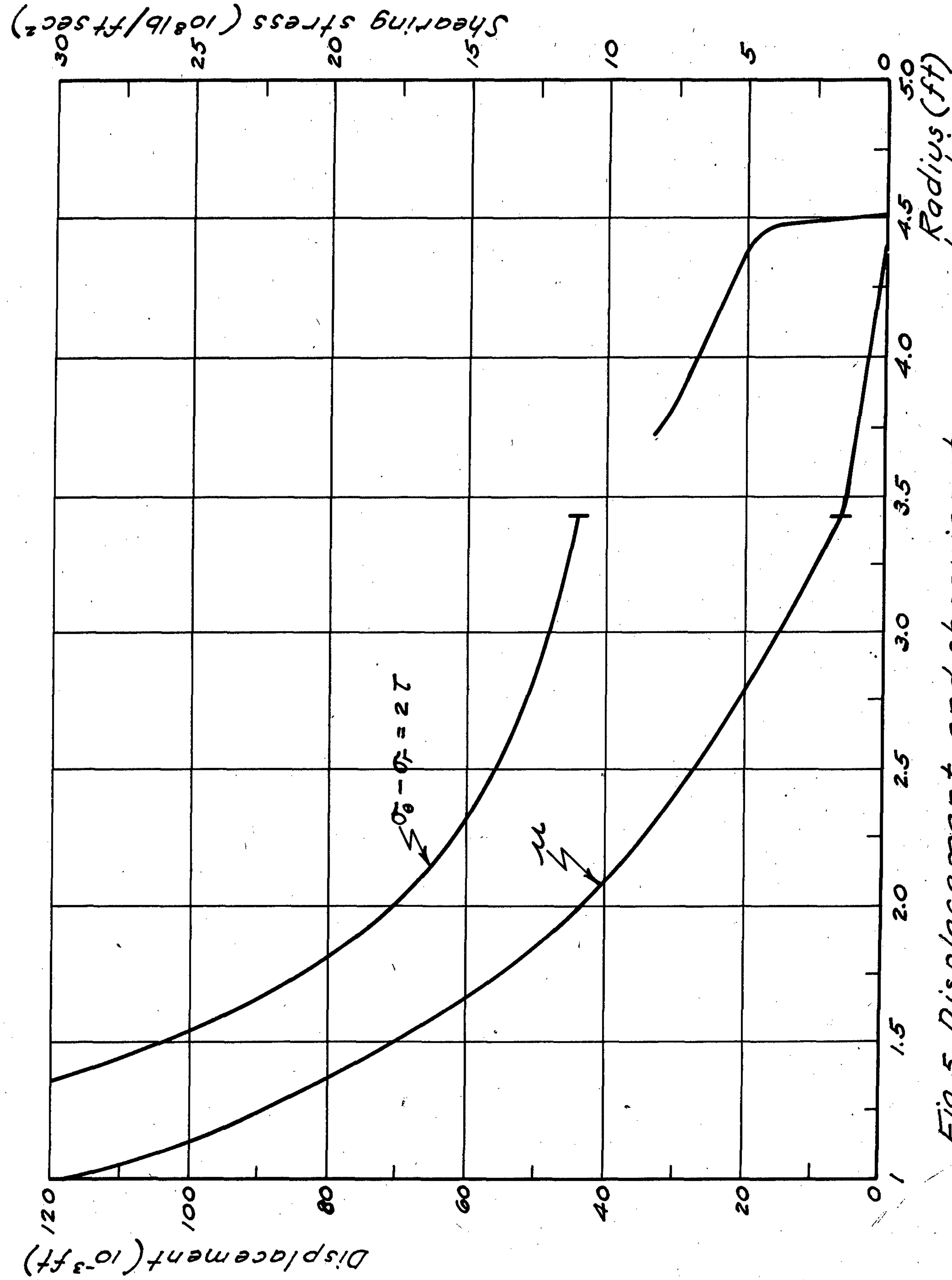

Fig. 5. Displacement and shearing stress produced in a thick plate by a uniformly expanding cylinder. Expansion has been going on for 2.3×10^{-4} sec.

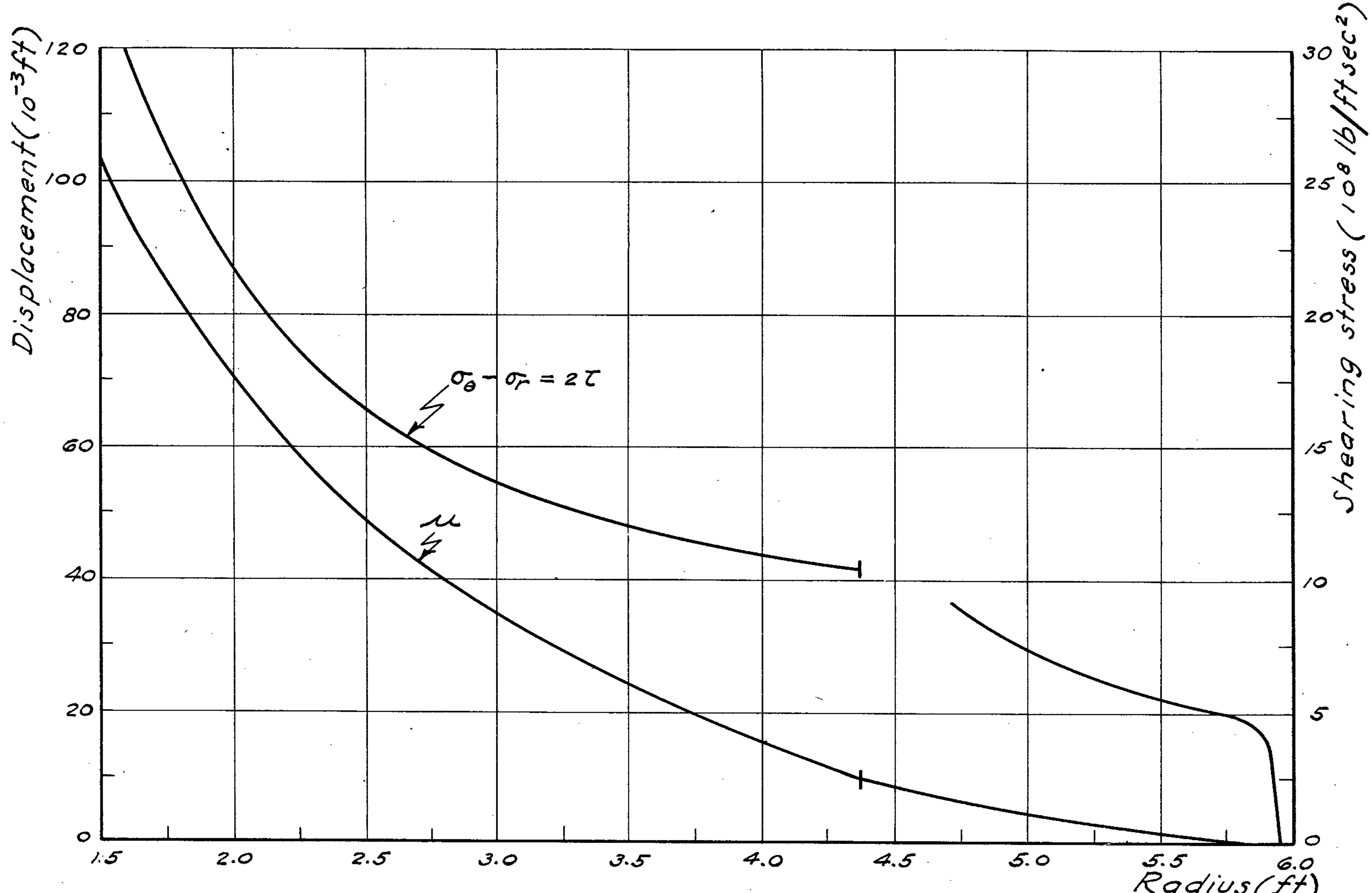

Fig. 6. Displacement and shearing stress produced in a thick plate by a uniformly expanding cylinder. Expansion has been going on for 3.3×10^{-4} sec.

$r - b = c_2(t - 2\Delta)$. In the elastic region the third diagonal row is started at (9,7) rather than at (10,8). Since the changes which are made along the edge of the plastic region will affect displacements at later times, it is important that the shearing strain at the edge of the plastic region be evaluated frequently as one moves down the diagram in order to avoid useless calculation.

In our numerical work we have used the following values for the constants which appear:

$c_1 = 14{,}910$ ft/sec, $\gamma_0 = 8.54 \times 10^{-3}$,

$c_2 = 11{,}240$ ft/sec, $G/F = 6.94$,

$c_2/c_1 = 0.754$, $b = 1$ ft,

$G = 5.40 \times 10^{10}$ lb/ft sec^2 $v_0 = 518$ ft/sec,

$$2\gamma_0(G - F)/\rho = 1.64 \times 10^6 \text{ ft}^2/\text{sec}^2.$$

The foregoing values of the constants are not appropriate for armor plate; however, the qualitative nature of the waves produced does not depend on the numerical values of the constants. The wave velocities are both about 25 percent too small. The stresses that we calculate will therefore be somewhat too large. From $t = 0$ to $t = 1.1 \times 10^{-4}$ sec the time interval $\underline{\Delta}$ used is 10^{-5} sec. From $t = 1.1 \times 10^{-4}$ to $t = 3.3 \times 10^{-4}$ sec the interval used is 2×10^{-5} sec. The large time interval is chosen so that the change in the displacement in going from one point in a diagonal row to the next is never more than 16 percent of the displacement. The average change is much less, amounting to 5 percent of the displacement. In Figs. 4, 5 and 6 we have plotted the displacements and the shearing

stresses as a function of the radius at times $t = 1.1 \times 10^{-4}$ sec, $t = 2.3 \times 10^{-4}$ sec and $t = 3.3 \times 10^{-4}$ sec. Figures 4, 5 and 6 show that the shearing strain decreases as the radius r increases. Initially there is a sizeable discontinuity in the shearing strain at the edge of the plastic region, but this becomes less marked as time passes.

5. Future program

From what has been said it is clear that the investigation of the propagation of plastic deformation in cylinders is by no means complete. In this section we shall mention a few points that should be investigated.

The derivation of the wave equations in this report indicates that there should be a minimum wave velocity for cylindrical waves of deformation. This minimum wave velocity depends chiefly on the compressibility and the density of the material. In the case of the propagation of plastic deformation in a wire no such minimum velocity is found. It would be interesting to determine whether such a minimum wave velocity can be found experimentally.

In the penetration of armor plate the strains near the axis of the shell are not infinitesmal. Therefore, the effect of these finite strains on the propagation of plastic deformation near the axis of the expanding cylinder should be considered theoretically.

A solution of the problem in the form of a series or some approximate function would be of much use since one could easily determine the result of a change in the physical constants of the material under consideration. For this reason it would be well to attempt an analytical solution of the case where the cylinder contains

both elastic and plastic waves. The approximate solution in the case where only elastic waves are present should furnish useful information for this work.

The experience gained in working with a two-dimensional model of the penetration problem might conceivably point the way to a numerical solution of the three-dimensional problem of the penetration of armor plate.

In the numerical solution given in this report the displacement at the surface of the expanding cylinder can be taken to be any desired function of the time. Thus, one could investigate the "secondary wave"[12/] which is sent out when the displacement at the surface of the expanding cylinder reaches its maximum value. To do this one would simply carry through a numerical solution using the wave equations, Eqs. (4), (8) and (13), together with the boundary condition,

$$u(b) = 0, \qquad \text{for } t < 0;$$
$$u(b) = v_0 t, \qquad \text{for } 0 < t < T;$$
$$u(b) = v_0 T, \qquad \text{for } T < t.$$

It is obvious, of course, that more detailed numerical calculations should be carried out using constants appropriate for armor plate. However, it would seem that such numerical calculations should be made after one knows how to treat the finite strains near the shell.

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2. In this calculation we used a stress-strain curve made up of two straight lines. The constants that fix the straight lines are given immediately after Eqs. (5) and (15) of this paper. The calculation was made using the theory proposed by von Kármán (see reference 10).

3. G. I. Taylor, "Notes on Bethe's theory of armor penetration, I, Static penetration," R. C. 279. (Sept. 1942).

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4. Bethe, reference 1, p. 73, or Love, The mathematical theory of elasticity (Cambridge Univ. Press, ed. 4, 1927), p. 288, Eqs. (42).

5. S. Timoshenko, Theory of elasticity (McGraw-Hill, ed. 1, 1934), p. 62, Eqs. (45) and (46).

6. Bethe, reference 1, p. 8, or Nadai, Plasticity (McGraw-Hill, 1931), p. 61.

7. Bethe, reference 1, Sec. 12, or Nadai, Plasticity, p. 73.

8. We have assumed that K is a constant. To check this we have calculated the change in K which occurs during the compression of cylindrical copper crushers in the apparatus of Seitz, Lawson, and Miller [NDRC Report A-63 (OSRD No. 619)]. The largest stress produced in their static experiment was 5.6×10^4 lb/in.^{2} Using data on the change of K produced by a change of volume [see F. Birch, J. App. Phys. 9, 279 (1938)] we find that K increases by 1.1 percent during such a test. Other metals yield very similar results.

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10. T. von Kármán, On the propagation of plastic deformation in solids, NDRC Report A-29 (OSRD No. 365).

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GLOSSARY

b	The initial radius of the cylindrical hole in a thick plate of infinite extent (Secs. 3 and 4).
c_1	Wave velocity in the elastic region.
c_2	Wave velocity in the plastic region.
D	Slope of the stress-strain curve in the plastic region.
E	Young's modulus.
F	Modulus in plastic region corresponding to shear modulus in elastic region; $F = d/2(1 + \alpha)$. SHOULD BE CAPITAL: D, see p. 8
f	Slope of stress-strain curve, some function of the largest shearing strain.
f'	Derivative of function f with respect to its argument.
G	Modulus of rigidity; $G = E/2(1 + \nu)$; also referred to as the shear modulus.
K	Modulus of volume expansion.
r, θ	Plane polar coordinates.
T	Time during which expansion of hole takes place (Sec. 3).
t	Time.
u	Displacement.
v_0	Radial velocity of material at the surface of the expanding cylindrical hole (Sec. 3).
α	Defined by $2\alpha = 1 - D/3K$
Δ	Time interval.
$\gamma_{r\theta}, \gamma_{rz}, \gamma_{\theta z}$	Shearing strains.
γ_0	Shearing strain when material begins to deform plastically.
γ_1	Shearing strain at point of decreasing strains.
$\varepsilon_r, \varepsilon_\theta, \varepsilon_z$	Tensile strains in the directions r, θ and z.
$\varepsilon_{r_0}, \varepsilon_{\theta_0}, \varepsilon_{z_0}$	Strains present when the material begins to deform plastically.

$\varepsilon_{r_1}, \varepsilon_{\theta_1}, \varepsilon_{z_1}$	Strains present at the time when the stresses in the vicinity of the cylindrical hole begin to decrease.
λ	The yield stress obtained in tensile tests made on long thin wires.
ν	Poisson ratio.
ρ	Density of the material.
$\sigma_r, \sigma_\theta, \sigma_z$	Tensile stresses in the directions $\underline{r}$ and $\underline{\theta}$.
$\sigma_{r_0}, \sigma_{\theta_0}, \sigma_{z_0}$	Stresses required to initiate plastic deformation.
$\sigma_{r_1}, \sigma_{\theta_1}, \sigma_{z_1}$	Stresses present at the time when the stresses in the vicinity of the cylindrical hole begin to decrease.

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ABSTRACT:

In the present paper it is assumed, in accord with Bethe's model for armor penetration, that the plastic deformation produced during armor penetration is similar to that produced by an expanding cylinder. In the first portion of the paper, the relations between the stresses and the strains are considered, and the wave equations which govern the motion of the material when it is rapidly deformed are derived. It is found that in the case of the thick plate, an elastic wave diverges radially from the expanding cylinder, that this is followed by a plastic wave, and that the elastic and plastic wave velocities do not differ very much. In the second portion of the paper, an approximate expression for the displacement is obtained for the case where the deformation is elastic. In the last section of the paper, a method of numerically integrating the wave equations is given.

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